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EE 272 - Dynamics of Lasers
Homework 1 : Laser bifurcation
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From the rate equations, we have seen in class that the bifurcation scenario occurring at $A = 1$ is a so-called transcritical bifurcation. In fact, such a bifurcation points out two steady state solutions, one stable and one unstable, exchanging their stability. We have proved that the OFF state is stable (unstable) below (above) threshold and the ON state is stable (unstable) above (below) threshold. However, the intensity I of the laser output needs to satisfy the physical constraint $I > 0$ and therefore the ON solution can not exist below threshold.
We may avoid this constraint by reformulating the rate equations in terms of the laser field rather than its intensity such as :

$$\frac{dE}{dT} = \frac{GNE}{2} - \frac{E}{2T_c} \tag{1}$$

Question 1 Assuming $\epsilon \equiv \sqrt{2GT_1}E$ and $I \equiv \epsilon^2$, write $d\epsilon/dt$ in a dimensionless form.
Using the aforementioned change of variables, we have,

$$\frac{dE}{dT} = \frac{1}{\sqrt{2GT_1}}\frac{d\epsilon}{dT} \tag{2}$$

and,

$$\frac{d\epsilon}{dT} = \frac{GN\epsilon}{2} - \frac{\epsilon}{2T_C} \tag{3}$$

Assuming that $t = T/T_c$ and $D = GNT_c$, we get,

$$\frac{d\epsilon}{dt} = \frac{1}{2}\epsilon\,(GNT_c - 1) = \frac{1}{2}\epsilon\,(D-1) \tag{4}$$

Question 2 Determine the new steady states and their linear stability properties.
The rate equation for carriers is expressed as,

$$\frac{dD}{dt} = \gamma\,[A - D(1+I)] = \gamma\left[A - D(1+\epsilon^2)\right] \tag{5}$$

The steady-states are given by $\frac{d\epsilon}{dt} = 0$ and $\frac{dD}{dt}$ that is,

$$\frac{1}{2}\epsilon_s(D_s - 1) = 0 \tag{6}$$

$$\gamma[A - D_s(1 + |\epsilon_s|^2)] = 0 \tag{7}$$

We conclude that the steady-states conditions are given by :
• If $\epsilon = 0$ (OFF state), then $D_s = A$

• If $\epsilon \neq 0$ (ON state), then $\epsilon_s = \pm\sqrt{A-1}$ with $A > 1$

In order to investigate the stability, let us assume a perturbation of the steady-states,

$$\epsilon = \epsilon_s + u \tag{8}$$

$$D = D_s + v \tag{9}$$

leading to,

$$\frac{du}{dt} = \frac{1}{2}(\epsilon_s + u)(D_s - 1 + v) \tag{10}$$

$$\frac{dv}{dt} = \gamma[A - (D_s + v)(1 + (\epsilon_s + u)^2)] \tag{11}$$

Assuming $\epsilon_s(D_s - 1) = 0$ and $uv << 1$, we get the following differential equations

$$\frac{du}{dt} = \frac{1}{2}[(D_s - 1)u + \epsilon_s v] \tag{12}$$

$$\frac{dv}{dt} = -\gamma[2D_s\epsilon_s u + (1 + \epsilon_s^2)v] \tag{13}$$

The Jacobian matrix calculated near (ϵ_s, D_s) is given by,

$$J_{(\epsilon_s, D_s)} = \begin{bmatrix} \frac{1}{2}(D_s - 1) & \frac{1}{2}\epsilon_s \\ -2\gamma D_s\epsilon_s & -\gamma(1 + \epsilon_s^2) \end{bmatrix}$$

The eigenvalues are extracted by the solving $P(\lambda) = \det[J_{(\epsilon_s, D_s} - \lambda I_d]$ hence,

$$P(\lambda) = \det\begin{bmatrix} \frac{1}{2}(D_s - 1) - \lambda & \frac{1}{2}\epsilon_s \\ -2\gamma D_s\epsilon_s & -\gamma(1 + \epsilon_s^2) - \lambda \end{bmatrix}$$

leading to the polynomial,

$$P(\lambda) = \lambda^2 + \lambda[\gamma(1 + \epsilon_s^2) + \frac{1 - D_s}{2}] + [\gamma D_s\epsilon_s^2 + \frac{1}{2}\gamma(1 + \epsilon_s^2) - \frac{1}{2}\gamma D_s(1 + \epsilon_s^2)] \tag{14}$$

• OFF state with $\epsilon_s = 0$ and $D_s = A$, we find two real eigenvalues that are $\lambda_1 = \frac{1}{2}(A - 1)$ and $\lambda_2 = -gamma < 0$. We conclude that for $A < 1$ both are negative meaning that the solution is stable while it gets unstable for $A > 1$.
• ON state with $\epsilon \neq 0$ and $\epsilon_s = \pm\sqrt{A-1}$, we find two complex eigenvalues that are $\lambda_{1,2} = -\frac{\gamma A}{2} \pm i\sqrt{\gamma\epsilon_s^2}$ assuming $\gamma << 1$. Because the amplitude of the electric field is a real number, we must constraint $A > 1$. In such way, real parts of eigenvalues are negative leading to stable solutions for $A > 1$.

Question 3 Sketch the bifurcation diagram. Conclude.
The linear stability analysis of the steady state solutions unveils that the OFF solution is stable (unstable) below (above) threshold while both ON solutions are stable in their whole domain of existence ($A > 1$). The pitchfork bifurcation is represented in Figure 1. Below threshold, only the OFF solution is possible. Beyond threshold, two new solutions corresponding to the ON state are available. In this case, the bifurcation is called supercritical since the new solutions overlap the unstable basic solution.

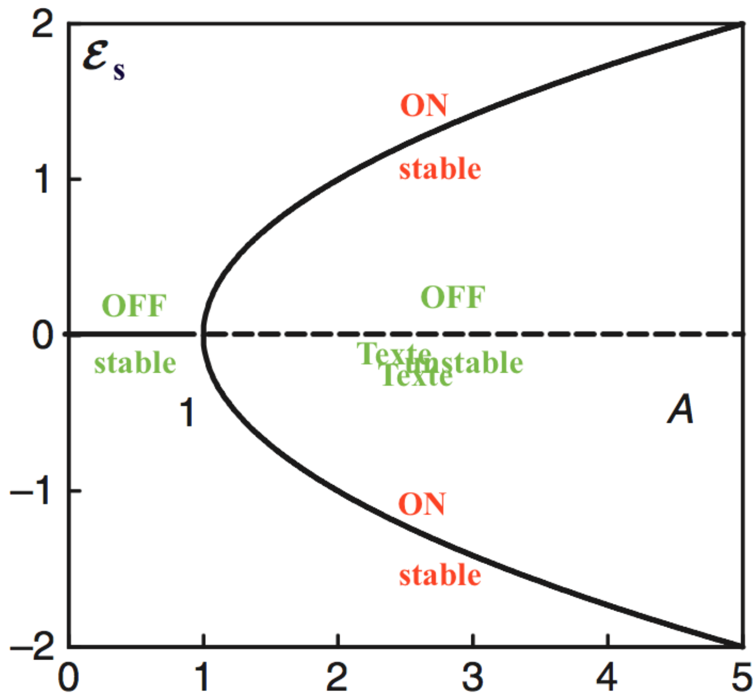

Figure 1 – Representation of the pitchfork bifurcation for the laser electric field.